

**BOUNDS FOR TREASURY BOND FUTURES
PRICES AND EMBEDDED DELIVERY OPTIONS:
Theory and Empirical Analysis**

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ABSTRACT

The delivery options in Treasury bond futures are difficult to price. A recursive use of the lattice model is unavoidable for valuing such options, as Boyle (1989) demonstrates. As a result, an accurate valuation of these delivery options is very expensive. In this paper, we derive upper bounds of these embedded options using an American option pricing technique. These upper bounds are then translated into a lower bound of the Treasury bond futures price. The popular cost of carry model is shown to be an upper bound of the Treasury bond futures price. These bounds are then tested empirically for the period from January 1987 till December 1991 using a two-factor Cox-Ingersoll-Ross model of the term structure.

I. INTRODUCTION

The delivery options in Treasury bond futures are generally known as the quality option and three timing options: the accrued interest option, the wild card option, and the end-of-month option. The quality option gives the short the right to deliver any eligible bond (no less than 15 years to maturity or first call) and various timing options give the short the flexibility of making the delivery any time in the delivery month. The end-of-month option refers to the deliveries occurring at the last 7 business days in the delivery month when the futures market is closed to trading. For the remaining days of the delivery month, the wild card option refers to about 6 hours from 2:00 p.m. to 8:00 p.m. (Chicago time) when the futures market is closed while the accrued interest option refers to the period from 7:20 a.m. to 2:00 p.m. when both futures and its underlying bond markets are open.

Delivery options in T bond futures are difficult to price. A recursive use of the lattice model is unavoidable for valuing such options, as Boyle (1989) demonstrates. As a result, an accurate valuation of these delivery options is very expensive. The goal of this study is therefore to derive fast bounds of these options and further to provide a lower bound for the T bond futures price. These bounds can be computed quickly and provide a crude conservative estimate of the T bond futures price.

An early discussion of the valuation of the quality option appears in Cox, Ingersoll, and Ross (1981) in which they state that their valuation can be applied to futures with the quality option when the single spot bond price is replaced with the minimum from the deliverable set. Jones (1985) argues that although there are multiple bonds eligible, bonds with extremely high and low durations are the ones to be delivered. His argument holds if the yield curve is flat. If the yield curve is not flat, then durations of different maturity bonds are not directly comparable and therefore his extreme duration rule fails. Hemler (1988) uses Margrabe's (1978) exchange option formula to price the quality option but the pricing formula becomes intractable as the number of deliverable bonds increases. Carr (1988) was the first to use factor models to price the quality option and Carr and Chen (1996) extend the Carr model to include a second factor. Ritchken and Sankarasubramanian (1992) use the Heath-Jarrow-Morton (1992) framework to find the quality option value. Livingston (1987) analyzes the quality option on the forward contract.

Timing options in general have no closed form solutions and are therefore studied with lattice methods. Kane and Marcus (1986) lay out a general framework for analyzing the wild card option. In their analysis, discounting is not considered in the wild card

period. Broadie and Sundaresan (1987) develop a lattice model to value the end-of-month option. Their focus is strictly on the futures price in the end-of-month period. Boyle (1989) uses a two period model to show that the timing option could have a significant impact. His analysis assumes constant interest rates and does not directly apply to T bond futures.

Empiricists in general agree that the quality option has a non-trivial value¹. However, unlike the evidence for the quality option, the evidence for the timing option is not so clear. This is because most studies do not distinguish between the quality option value and the value from other timing options, let alone values among various timing options². In this paper, we derive an upper bound for each timing option separately. Combining these upper bounds, we can establish a lower bound for the futures price. Since the cost of carry model provides an upper bound for the futures price, we can bound the futures price. We then provide empirical results to show that these bounds are very tight — about 2% up and below the futures price.

The paper is organized as follows. The next section provides the theoretical analysis and derives upper bounds for various delivery options and further shows the upper and lower bounds of the futures price. Section III contains an empirical study where a two factor equilibrium term structure model is explicitly implemented to investigate the magnitude of each timing option. Finally, the paper is concluded in Section IV.

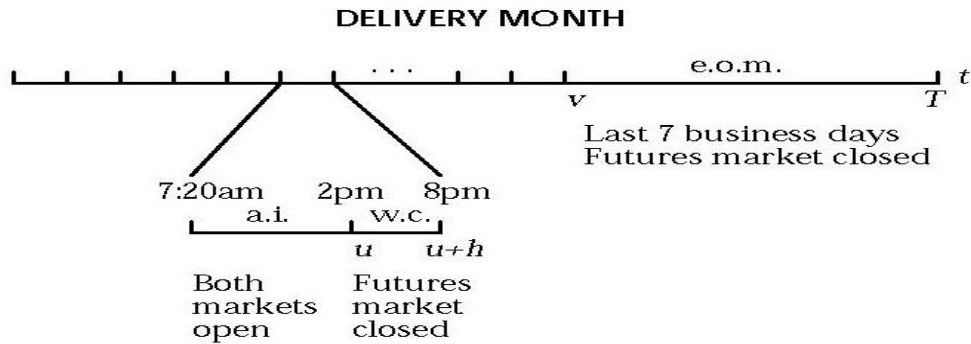
II BOUNDS OF DELIVERY OPTIONS AND FUTURES PRICE

The delivery options are known as the quality option and three timing options. The short of the futures contract has the right to choose the cheapest bond to deliver as well as to deliver at any time in the delivery month. The short can make a delivery even when the futures market is closed. At the end of the delivery month, for 7 business days, the futures market is closed but the short can still make a delivery. This is understood as the end-of-month timing option. For the remaining about 15 business days in the delivery month, the short can deliver either between 7:20 a.m. and 2:00 p.m. when both the futures market and the underlying bond market are open or till 6 hours after the futures market is closed. The former timing option is called the accrued interest timing option and the

¹ See, for example, Carr and Chen (1996), Kilcollin (1982), Benninga and Smirloc (1985), Kane and Marcus (1986), and Hedge (1990).

² See, for example, Arak and Goodman (1987), Hedge (1988), Gay and Manaster (1986).

latter timing option is also known as the daily wild card play. The following picture graphically explains various timing options.



The last 7 business days is the end-of-month period. Throughout the paper we use v for the starting time and T for the ending time of this period. For the rest of the delivery month, there are two sections of each day, the accrued interest period and the wild card period. The wild card period for each day will be labeled u and $u+h$. The notation and symbols used in the paper are also summarized as follows:

- $\Phi(t)$ = Futures price with all delivery options
- $\Phi_i(t)$ = Futures price of the i th bond
- $\Psi_i(t)$ = Forward price of the i th bond
- $a_i(t)$ = Accrued interest of the i th bond
- $P_{i,T}$ = Discount bond price at time t of \$1 at time T
- $Q_i(t)$ = Coupon bond price of the i th bond
- q_i = Conversion factor of the i th bond

Before we start our analysis, we need Jamshidian's separation theorem (1987) and his definition of the forward measure.

Theorem 1 (Jamshidian 1987)

Let P be the price of a pure discount bond at time t delivering \$1 at some future date T and follow the dynamics:

$$\frac{dP_{i,T}}{P_{i,T}} = r_i dt + b_{i,T} d\hat{W}_i$$

where r is the instantaneous risk-free rate, b is a maturity dependent volatility parameter, and $d\hat{W}$ is Wiener differential under the risk-neutral space. Then the forward measure is defined as:

$$\frac{dP_{i,T}}{P_{i,t}} = (r_i - b_{i,T}^2)dt + b_{i,T} d\bar{W}_i$$

where $d\bar{W}_i = d\hat{W}_i + b_{i,T} dt$. Any discounted expected payoff of an asset can be separated into a product of the pure discount bond price and the forward price of the asset, that is:

$$\hat{E}_t[\delta_{i,T} X(T)] = \hat{E}_t[\delta_{i,T}] \bar{E}_t[X(T)] = P_{i,T} \bar{E}_t[X(T)]$$

where $\delta_{i,T} = \exp\left(-\int_t^T r(w)dw\right)$ and $\bar{E}_t[X(T)]$ computes the forward price of X .

A simple proof of this theorem is given in an appendix while the original proof is available in Jamshidian (1987).

A. The Quality Option

In the absence of all timing options, the quality option gives the short a right to deliver the cheapest bond only at maturity, T , and the short receives the following payoff:

$$\text{Eq 1} \quad \max\{q_i \Phi(T) - Q_i(T)\}$$

Note that the accrued interests of both bond and futures contracts are equal and canceled. Since the delivery value of Eq 1 has to be identically 0 for all states, we can solve for the futures price at maturity as:

$$\text{Eq 2} \quad \Phi(T) = \min\left\{\frac{Q_i(T)}{q_i}\right\}$$

and today's futures price is merely a risk-neutral expectation of this payoff:

$$\begin{aligned}
 \Phi(t) &= \hat{E}_t \left[\min \left\{ \frac{Q_i(T)}{q_i} \right\} \right] \\
 \text{Eq 3} \quad &= \frac{\hat{E}_t[Q_1(T)]}{q_1} - \hat{E}_t \left[\max \left\{ \frac{Q_1(T)}{q_1} - \frac{Q_i(T)}{q_i} \right\} \right] \\
 &= \frac{\Phi_1(t)}{q_1} - \hat{E}_t \left[\max \left\{ \frac{Q_1(T)}{q_1} - \frac{Q_i(T)}{q_i} \right\} \right]
 \end{aligned}$$

Note $\Phi_1(t) = \hat{E}_t[Q_1(T)]$ is the futures price of the first bond with no option and $\Phi(t)$ is the futures price of the cheapest bond at maturity. This result has been shown previously by Carr (1988) and other authors. This equation says that the futures contract with the quality option is equivalent to a futures contract without the quality option (only bond 1 is eligible for delivery) with an exchange option held by the short.

Eq 3 is correct only if marking to market exists throughout the life of the futures contract. Unfortunately, in the last 7 business days of the delivery month, the futures market is not open and the futures contract is not marked to market. The futures price used for settlement in this period is the last settlement price at the beginning of the 7-day period. Since the futures price is already determined, the actual payoff at the last delivery day, T , is not necessarily 0. The short can actually gain or lose. To avoid arbitrage, the futures price at the beginning of the 7-day period should be set so that the expected present value of payoffs at maturity is 0. Under this circumstance, the futures price at the beginning of the 7-day period is a forward price, not a futures price. Formally, the futures price at the beginning of the end-of-month period, v , should be so set that:

$$\text{Eq 4} \quad \hat{E}_v \left[\delta_{v,T} \max \{ \Phi(v)q_i - Q_i(T) \} \right] = 0$$

where δ is the stochastic discount factor assumed to be strictly less than 1. Using Theorem 1, we can then write:

$$\text{Eq 5} \quad \bar{E}_v \left[\max \{ \Phi(v)q_i - Q_i(T) \} \right] = 0$$

and the futures price at time v can be written as:

$$\text{Eq 6} \quad \Phi(v) = \frac{\Psi_1(v)}{q_1} - \frac{1}{q_1} \bar{E}_v [\max\{Q_1(T) - Q_i(T) - K_i\}]$$

where $K_i = (q_1 - q_i)\Phi(v)$. Note that $\Psi_1(v) = \bar{E}_v[Q_1(T)]$ is the forward price of the first bond. The interpretation of this result is similar to that of Eq 3, except that the risk neutral measure is replaced by the forward measure defined in Theorem 1 and the futures price becomes the forward price. However, unlike Eq 3, the futures price at time v has no easy solutions, because it appears on both sides of the equation. This futures price has to be solved recursively using a lattice method, as suggested by Boyle (1989). That is, we first choose an initial value for the futures price at time v , calculate payoffs at various states at maturity T , and then work backwards along the lattice. We adjust the futures price until the discounted payoff computed from the lattice is 0. Once the futures price at time v is set, we can then travel back along the lattice and use risk neutral probabilities to find the futures at any time³

With the presence of the end-of-month timing option, the futures price computed by Eq 6 is an overestimate because the short has additional flexibility of choosing the best timing. If the short is allowed to delivery any time in this 7-day period, then we need to compare the expected present value of future payoffs with the current delivery value. Higher current delivery value will trigger early deliveries. This is very similar to the American option pricing methodology where the intrinsic value is compared by the expected present value of future payoffs.

B. The Accrued Interest Timing Option

The accrued interest timing option refers to the flexibility for the short to deliver the cheapest bond any time in the delivery month when both futures and spot markets are open. This is everyday from 7:20 a.m. to 2:00 p.m. (Chicago time) from the first day of the delivery month to right before the end-of-month period. Since the futures market is open, the futures contract is marked to market and deliveries can take place any time. As a result, the futures price can never be greater than the cheapest to delivery bond price. If the futures price were greater than the cheapest bond price, then deliveries would take

³ The futures price becomes a forward price also in each of the wild card periods. The recursive use the lattice model is required for the daily wild card periods.

place instantly. The short will sell the futures, buy the cheapest bond, make the delivery, and earn an arbitrage profit. Formally,

$$\text{Eq 7} \quad \Phi(t) > \min \left\{ \frac{Q_i(t)}{q_i} \right\}$$

$$\max \{ \Phi(t)q_i - Q_i(t) \} > 0$$

Therefore, the futures price in the period where both markets are open must be less than the cheapest to delivery bond price to avoid arbitrage. On the other hand, if the futures price is lower, one can long futures and short spot but the delivery will not occur because the short position of the futures contract will lose money if he makes a delivery. Consequently, the delivery will be postponed and there is no arbitrage profit to be made. If the futures price is always less than the cheapest to delivery bond price (adjusted by its conversion factor), the delivery payoff now is negative as opposed to 0 at the end. As a result, the short will never deliver until the last day. Consequently, the accrued interest timing option has no value. We restate this result in the following proposition.

Proposition 1

The accrued interest timing option without the wild card and end-of-month options has no value.



The existence of the other timing options will lower the current futures price, further reducing the incentive for the short to deliver early. We state this result in the following Corollary.

Corollary 1-1

The accrued interest timing option with the wild card and end-of-month options has no value.



While the accrued interest timing option is worthless, the timing options at the end-of-month and the wild card periods are valuable. When the futures market is closed, there is no marking to market in the futures market and the futures contract becomes a forward contract. Boyle (1989) has demonstrated that in a case of forward contracts timing

options will have value. We shall extend Boyle's analysis to stochastic interest rates so that we can evaluate T bond futures timing options.

C. The End-of-month Timing Option

Without the end-of-month timing option, we know that the futures price should be set according to Eq 6. With the end-of-month timing option, deliveries can occur any time in the end-of-month period as long as the current delivery payoff is more than the present value of the expected payoff.

For both quality and timing options to exist, any higher delivery payoff will replace the present value of the expected payoff, like the early exercise of an American option. The futures price at the beginning of the period needs to be so set that the expected discounted payoff is nil. As a result, if we can identify a function that is always greater than both the delivery payoff and the discounted present value, this function is guaranteed to have a positive present value at time v .

The upper bound function is given as follows:

$$\text{Eq 8} \quad \hat{E}_v \left[\max \left\{ \frac{1}{\delta_{v,T}} \Phi(v) q_i - Q_i(T) \right\} \right]$$

where \mathbf{d} is the stochastic discount factor which is assumed to be strictly less than 1^4 . This value is greater than the present value of the delivery payoff at any time $t \in [v, T]$:

$$\begin{aligned} \hat{E}_v \left[\max \left\{ \frac{1}{\delta_{v,T}} \Phi(v) q_i - Q_i(T) \right\} \right] &> \hat{E}_v \left[\max \left\{ \Phi(v) q_i - \delta_{v,T} Q_i(T) \right\} \right] \\ &> \hat{E}_v \left[\max \left\{ \Phi(v) q_i - \delta_{t,T} Q_i(T) \right\} \right] \\ \text{Eq 9} \quad &= \hat{E}_v \left(\hat{E}_t \left[\max \left\{ \Phi(v) q_i - \delta_{t,T} Q_i(T) \right\} \right] \right) \\ &\geq \hat{E}_v \left[\max \left\{ \Phi(v) q_i - Q_i(t) \right\} \right] \\ &> \hat{E}_v \left[\delta_{v,t} \max \left\{ \Phi(v) q_i - Q_i(t) \right\} \right] \end{aligned}$$

The second to last line of the above equation holds through using Jensen's inequality and $\hat{E}_t [\mathbf{d}_{t,T} Q_i(T)] < Q_i(t)^5$.

⁴ Note that $P_{i,T} = \hat{E}_t [\mathbf{d}_{t,T}]$ is the pure discount bond price of \$1.

⁵ Let a be the accrued interest. Then, by the martingale property of any traded asset, $\hat{E}_t [\mathbf{d}_{t,T} Q_i(T)] + P_{i,T} a_i(T) = Q_i(t) + a_i(t)$. Since the accrued interests are linear but discounting is not, we can show that with normal interest rate levels, the following relationship, $\hat{E}_t [\mathbf{d}_{t,T} Q_i(T)] < Q_i(t)$, holds.

Eq 9 shows that the proposed function is greater than the delivery at any time. For the proposed function to be an upper bound, we need to also show that the function has a higher value at an earlier time than at a later time. That is:

$$\text{Eq 10} \quad \hat{E}_v \left[\max \left\{ \frac{1}{\delta_{v,T}} \Phi(v) q_i - Q_i(T) \right\} \right] > \hat{E}_v \left[\hat{E}_i \left[\max \left\{ \frac{1}{\delta_{i,T}} \Phi(v) q_i - Q_i(T) \right\} \right] \right] \\ > \hat{E}_v \left[\max \left\{ \frac{1}{\delta_{i,T}} \Phi(v) q_i - Q_i(T) \right\} \right]$$

Since the function is an upper bound, its value at the beginning of the end-of-month period, v , should be positive:

$$\text{Eq 11} \quad \hat{E}_v \left[\max \left\{ \frac{1}{\delta_{v,T}} \Phi(v) q_i - Q_i(T) \right\} \right] > 0$$

This implies that the futures price should be bounded from below as follows:

$$\text{Eq 12} \quad \Phi(v) > \frac{\Phi_1(v) \Delta_{v,T}}{q_1} - \frac{\Delta_{v,T}}{q_1} \hat{E}_v \left[\max \left\{ Q_1(T) - Q_i(T) - \frac{1}{\delta_{v,T}} K_i \right\} \right] \\ > \frac{\Phi_1(v) \Delta_{v,T}}{q_1} - \frac{\Delta_{v,T}}{q_1} \hat{E}_v \left[\max \left\{ Q_1(T) - Q_i(T) - K_i \right\} \right]$$

where

$$K_i = (q_1 - q_i) \Phi(v) \text{ and} \\ \Delta_{v,T} = \frac{1}{\hat{E}_v[1/\delta_{v,T}]} < 1.$$

Note that the second inequality holds because d is strictly less than 1. Therefore, the right hand side of the above equation is a lower bound. If we express it differently,

$$\begin{aligned} \text{Eq 13} \quad & \frac{\Phi_1(v)\Delta_{v,T}}{q_1} - \frac{\Delta_{v,T}}{q_1} \hat{E}_v[\max\{Q_1(T) - Q_i(T) - K_i\}] \\ & = \frac{\Phi_1(v)}{q_1} - \left\{ \frac{\Phi_1(v)}{q_1} (1 - \Delta_{v,T}) + \frac{\Delta_{v,T}}{q_1} \hat{E}_v[\max\{Q_1(T) - Q_i(T) - K_i\}] \right\} \end{aligned}$$

then bracketed term of the above equation is an upper bound of the end-of-month option. We state this result in a following proposition.

Proposition 2

*The end-of-month timing option is bounded from above by:*⁶

$$\text{Eq 14} \quad \frac{\Phi_1(v)}{q_1} (1 - \Delta_{v,T}) + \frac{\Delta_{v,T}}{q_1} \hat{E}_v[\max\{Q_1(T) - Q_i(T) - K_i\}]$$

Since K is always positive, if we ignore the time value (i.e. $\Delta = 1$), then the upper bound can be written as:

$$\text{Eq 15} \quad \frac{1}{q_1} \hat{E}_v[\max\{Q_1(T) - Q_i(T)\}]$$

■

It is interesting to note that the end-of-month option has value even if there exists no quality option. When there is no quality option but the timing option is allowed, the delivery may occur early. The short always compares the delivery payoff $\Phi(v)q - Q(t)$ where $t \in (v, T)$ with the expected present value of the delivery payoff at maturity $\hat{E}_t[\delta_{t,T}(\Phi(v)q - Q(T))]$. We can show that:

$$\text{Eq 16} \quad \hat{E}_t[\delta_{t,T}(\Phi(v)q - Q(T))] > P_{t,T}\Phi(v)q - Q(t) < \Phi(v)q - Q(t)$$

⁶ This upper bound contains both the quality option and the end-of-month timing option values.

Since the direction of the inequality can go either way, it is likely that early deliveries can take place. This demonstrates that the timing option does have value even in the absence of the quality option. The difference between the first two terms is $P_{t,T}a(T) - a(t)$ where a is the accrued interest and the difference of the last two terms is $(1 - P_{t,T})\Phi(v)$. As a result, whether or not deliveries will occur early depends upon which effect is larger. This result should not be confused with the result from Boyle (1989) where the timing option is defined differently.

D. The Wild Card Timing Option

In addition to the end-of-month period where the futures market is closed but the bond market is open, there is a 6 hour period every day for about 15 days where the futures market is also closed. This is called the daily wild card timing option. The wild card option is different from the end-of-month option in that the futures market will reopen after the wild card period and the futures contract will be marked to market. If bond prices drop in the wild card period, given that the futures price is fixed, the short can benefit from delivering a cheaper bond. However, the short can equally benefit from the marking to market when the futures market reopens. As a result, the incentive for the short to deliver in the wild card period is minimal. It has to be case where the payoff from immediate delivery exceeds the expected present value of marking to market; or the short will not deliver early in the wild card period.

At the end of each wild card period, $u + h$, the short decides whether or not to deliver by comparing the payoffs between marking to market and delivery:

$$\text{Eq 17 } \max\{\Phi(u) - \Phi(u + h), \Phi(u)q_i - Q_i(u + h)\}$$

If $\Phi(u + h)$ is small, then the marking to market payoff will outweigh the early delivery payoff and the wild card option will have no value. As a result, if we substitute a low enough value for $\Phi(u + h)$, we can always eliminate the wild card value. This is a useful result because if the upper bound of the end-of-month option has already reduced the futures price at the beginning of the end-of-month period to a low enough value, then this end-of-month upper bound includes not only the value of the end-of-month option but all the wild card options. As we shall present our empirical results in the next section, this is precisely the case for the 87-91 period.

It is interesting to note that the wild card option is completely induced by the conversion factor. If there were no conversion factor, the wild card option would have no value. Suppose there were no conversion factor. Then the payoff of the delivery at any time, $t \in (u, u + h)$, in the wild card period would be:

$$\text{Eq 18 } \max\{\Phi(u) - Q_i(t)\}$$

and the marking to market value at the end of the period is:

$$\text{Eq 19 } \Phi(u) - \Phi(u + h) \geq \Phi(u) - \min\{Q_i(u)\} = \max\{\Phi(u) - Q_i(u + h)\}$$

which is always greater than the delivery payoff at the end of the wild card period. The expected present value taken at time t of this payoff is always greater than the delivery payoff at time t (ignoring time value):

$$\text{Eq 20 } \hat{E}_t[\max\{\Phi(u) - Q_i(u + h)\}] \geq \max\{\Phi(u) - \Phi_i(t)\} \approx \max\{\Phi(u) - Q_i(t)\}$$

Therefore, the wild card option has no value. Note that when the conversion factor did not exist, the futures price would be set differently. And this different futures price will eliminate all incentives of early deliveries in the wild card period.

E. The Cost of Carry Model

After the lower bound of the futures price been derived, in the next proposition, we show that the cost of carry model provides an upper bound for the futures price. The well-known cost of carry formula is the following:

$$\text{Eq 21 } \Phi_{\text{occ}}(t) = \frac{[Q_*(t) + a_*(t)] / P_{t,T} - a_*(T)}{q_*}$$

where Q_* , q_* , and a_* are quoted price, conversion factor, and accrued interest of the cheapest bond at time t . Rearrange terms to get:

$$\begin{aligned}
\Phi_{\text{cost}}(t) &= \frac{\hat{E}_t \left[\frac{\delta_{i,T}}{E_{i,T}} (Q_*(T) + a_*(T)) \right] - a_*(T)}{q_*} \\
\text{Eq 22} \quad &= \bar{E}_t \left[\frac{Q_*(T)}{q_*} \right] = \bar{E}_t \left[\min \left\{ \frac{Q_i(T)}{q_i} \right\} \right] \\
&> \hat{E}_t \left[\min \left\{ \frac{Q_i(T)}{q_i} \right\} \right]
\end{aligned}$$

As we can see, the cost of carry model is equal to a *forward* expectation of the payoff. The futures price without the timing options is a *risk-neutral* expectation of the payoff (see Eq 3). The last inequality is obtained due to the following:

$$\text{Eq 23} \quad \text{cov}_t \left[\delta_{i,T}, \min \left\{ \frac{Q_i(T)}{q_i} \right\} \right] > 0$$

This is easy to see because when r increases (decreases), both discount factor, d , and all bonds, Q 's, decrease (increase), and the sign of the covariance is therefore positive. Note that the futures price without timing options is already an upper bound, the cost of carry model used by practitioners is a more conservative upper bound of the futures price. We state the result in the following proposition.

Proposition 5

The futures price is bounded from above by the cost of carry model

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III EMPIRICAL STUDY

A. Methodology

In this section, we empirically examine the magnitude of each bound using a two factor CIR model. We use the two factor model of the following kind:⁷

⁷ This two factor model is adopted by a number of authors. See Chen and Scott (1992), Turnbull and Milne (1992), Langetieg (1980), Hull and White (1990)

$$\text{Eq 24 } r = y_1 + y_2$$

where each factor follows a square root process:

$$\text{Eq 25 } dy_{jt} = (\mathbf{k}_j \mathbf{q}_j - (\mathbf{k}_j + \mathbf{l}_j) y_{jt}) dt + \mathbf{s}_j \sqrt{y_{jt}} d\hat{W}_{jt}$$

where $j=1,2$. The parameters \mathbf{k}, \mathbf{q} , and \mathbf{d} are constants. The parameter \mathbf{l} is a constant under log utility. This two factor model has been estimated by Chen and Scott (1993) with a weekly data set from 1980 to 1988. Their two factor model fits the yield curve reasonably well (for both in sample, 1980-88, and out of sample, 1989-91, periods). Three month, six month, five year, and the longest maturity Treasury issues are used to estimate the parameters for the two factor model as follows:

$$\begin{aligned} \kappa_1 &= 1.8341, \theta_1 = 0.05148, \sigma_1 = 0.1543, \lambda_1 = -0.1253 \\ \kappa_2 &= 0.005212, \theta_2 = 0.03083, \sigma_2 = 0.06689, \lambda_2 = -0.0665 \end{aligned}$$

We choose the bond with the largest conversion factor as our primary bond to deliver and calculate its futures price using the two factor version of the CIR model (1981). The quality option represents the option for the short to exchange a cheaper bond for this bond at delivery. Various timing options give the short additional flexibility of choosing the best timing. Although various bounds can be calculated using bivariate integrals, we choose to use a lattice model for those bivariate integrals are complex and not necessarily more efficient.

To calculate the upper bound value for the end-of-month option for any given time t_0 prior to v , we need to calculate the following expectation:

$$\text{Eq 26 } \hat{E}_{t_0} \left[\frac{\Phi_1(v)}{q_1} (1 - \Delta_{v,T}) + \frac{\Delta_{v,T}}{q_1} \hat{E}_v [\max\{Q_1(T) - Q_i(T) - K_i\}] \right]$$

The lower bound considering only the end-of-month option is therefore:

$$\text{Eq 27 } \hat{E}_{t_0} \left[\frac{\Delta_{v,T} \Phi_1(v)}{q_1} - \frac{\Delta_{v,T}}{q_1} \hat{E}_v [\max\{Q_1(T) - Q_i(T) - K_i\}] \right]$$

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Like any lattice pricing approach, we need to jump to the end of the lattice which is the end of the delivery month, T . At this terminal date, we need to compute the payoff, $\max\{Q_1(T) - Q_i(T) - K_i\}$, for all the states. Note that the strike price K contains $\Phi(v)$ the futures price at time v , which is not observable at time T . We use the futures price with the quality option as an approximation. As we shall see later, this approximation works very well. As a result, we need to first calculate $\hat{E}_v[\min\{Q_i(T)/q_i\}]$ using the lattice and then use the result to replace $\Phi(v)$ in K . Other than calculating the risk neutral expected value of the payoff, we also need the first bond price $Q_1(T)$ and the discount factor d to compute $\hat{E}_v[Q_1(T)] = \Phi_1(v)$ and $\Delta_{v,T}$. After all values are computed for time v , we then work backwards along the lattice till date t_0 . Since the bond prices at time T need to be quoted prices, we need to adjust the CIR formula values by the accrued interest for all the bonds at all the states.

Between the beginning of the end-of-month period, v , and the end of the last wild card period, $u+h$, is one accrued interest period. In this period, there is no option value to be computed and marking to market suggests that the futures price should be calculated directly from the risk neutral expectation.

As noted earlier, the wild card value can be ignored if the lower bound of the futures price at the beginning time of the end-of-month period, v , is already low enough. That is, if we use the lower bound for the end-of-month option, Eq 27, to substitute for $\Phi(u+h)$, the loss of the wild card value is translated into the end-of-month option. In other words, we can efficiently incorporate the wild card value into the lower bound for the end-of-month option. If the wild card value is eliminated completely by this substitution, then the lower bound for the end-of-month option becomes a lower bound for both end-of-month and wild card options. As we shall see below (subsection C), this is indeed the case for the period we examine.

Finally, the cost of carry model of Eq 22 is computed to compare with the futures price with only the quality option, i.e. Eq 3, and the actual futures price.

B. Data

The period of our study is from January, 1987 through December, 1991. We select weekly futures prices that have 6 weeks to 4½ months to maturity from the CBOT daily price data set. Since we focus on the futures contracts from March 1987 through December 1991, (20 contracts in total), our weekly observations start 1/8/87 and end 10/30/91.

The cost of carry model requires the knowledge of all deliverable bonds at the trade date. We collect all deliverable bonds from the *Wall Street Journal* for all the trade dates. We use the average of the bid and ask for the bond price. We also collect the three-month T bill rates for the cost of carry model. There are about 30 bonds for any given trade date.. Conversion factors are computed by the CBOT formula.

We assume no gap between the close of the bond market for any given day and the open of the futures market in the next morning. As a result, in order to correctly date all the timing periods in the lattice, we have to count the number of trading days. As been pointed out previously, there are about 22 trading days in a month. The last 7 days attribute to the end-of-month period and each of the remaining 15 days has about 6 hours for the day period where both bond and futures markets are open and another about 6 hours for the night period where only bond market is open. In order to accurately calculate various timing option values, the time to maturity in this study is not measured by calendar days but by business days⁸ Accurate day count is necessary because we need to calculate expectations at various times.

C. Results

Table 1 presents results in averages for the 20 contracts (8703 through 9112) studied in the paper. The first 3 columns of Table 1 present actual futures prices, lower bound futures prices using Eq 27 which considers only the end-of-month option, and upper bound futures prices using Eq 21 which is the cost-of-carry model. The average of the whole period is given at the bottom of the table. The cost of carry model is on average 2% higher than the actual futures price while the futures price with the end-of-month bound is 2% lower than the actual futures price. Daily prices of these three series are plotted in Figure 1. Since the futures price with only the quality option should be a tighter upper bound, we report this result using Eq 3 in column 4. It is seen that the futures price with only the quality option not only provides a tighter upper bound, it also approximates the actual futures price amazingly well. For all 20 contracts together, the average futures price with the quality option is 92.9091 which is less than 50 basis points higher than the average actual futures price. This result supports Carr and Chen (1996) in which the value of the quality option should explain most of the total delivery option value. It also supports the evidence that the cost-of-carry model is insufficient to explain the delivery option value.

⁸ That is, we do the business day count between trade day and the last day of the delivery month and assume 252 trading days for a given year.

Since the true futures price contains all embedded options, the total value of timing options can be implied by subtracting the actual futures price from the futures price with the quality option, i.e., subtracting (1) from (4). The results are reported in column 5. As we have argued, this value is quite small. Nonetheless, an average of 70 basis points is not a negligible quantity.

The end-of-month option bound values using Eq 26 are given in column 6. This value includes both the quality option and the timing option values. It is difficult to separate these two values because there is no consistent way to measure the quality option¹⁰. It is seen in Figure 1 that the lower bound for the futures price provided by this upper bound is conservative enough to include all daily wild card values. And the bound is as tight as the cost of carry model, about 2% on average lower than the actual futures price.

The last column reports wild card bound values. It is clear that if we can use a lower value for $\Phi(u + h)$, then the wild card option is included in the end-of-month bound.

IV CONCLUSION

In this paper, we derive lower and upper bound formulas for the Treasury bond futures price. The lower bound of the futures price is obtained by summing all upper bounds for the delivery options. The cost of carry model is found to be an upper bound of the futures price. These bounds provide investors with efficient range of how much futures prices can move. In a sample period of 1987-1991, the cost of carry model is found to be 2% above the actual futures price and the lower bound is found to be 2% below. A tighter upper bound which is the futures price with the quality option is found to approximate the actual futures price extremely well. On average, the approximation is above the actual futures price by only 50 basis points.

⁹ The futures price with the quality option sometimes is less than the actual futures price.

In this case, the timing option value is recorded as 0.

¹⁰ Carr and Chen (1996) measures the quality option value by looking at the difference between column 3 and column 4 in the Table. The quality option value, on the other hand, can be defined as the difference between the futures price without the quality option and the futures price with the quality option. Then, there is more than one measure for the quality option.

As opposed to recursively using the lattice model to compute the true futures price, the bounds provided in the paper can be computed quickly and accurately. Thus, these bounds can provide traders with a useful approximation of the true futures price.

APPENDIX

Theorem 1: Forward Measure and Separation Theorem

The separation theorem states that:

$$(A1) \quad \hat{E}_t[\delta_{t,T} X(T)] = P_{t,T} \hat{E}_t \left[\frac{\delta_{t,T}}{P_{t,T}} X(T) \right] = P_{t,T} \bar{E}_t[X(T)]$$

Since $\hat{E}_t[\mathbf{d}_{t,T} X(T)] = X(t)$, the forward expectation gives the forward price. From the above change of measure, it is clear that the Radon-Nykodim derivative is:

$$(A2) \quad \zeta = \frac{\delta_{t,T}}{\hat{E}_t[\delta_{t,T}]}$$

For notational convenience, we use subscripts for partial derivatives and move time indexes inside parentheses. Using Ito's lemma on the log of the bond price $P(T, T) = 1$ to get:

$$\begin{aligned} 0 &= \ln P(T, T) \\ &= \ln P(t, T) + \int_t^T \frac{1}{P(w, T)} (P_w(w, T)dw + P_r(w, T)dr + \frac{1}{2} P_{rr}(w, T)(dr)^2) dW(w) \\ &\quad - \int_t^T \frac{1}{2} \left(\sigma(r, w) \frac{P_r(w, T)}{P(w, T)} \right)^2 dw \\ &= \ln P(t, T) + \int_t^T \frac{1}{P(w, T)} (P_w(w, T) + \hat{\mu}(r, w)P_r(w, T) + \frac{\sigma^2(r, w)}{2} P_{rr}(w, T)) dw \\ &\quad + \int_t^T \sigma(r, w) \frac{P_r(w, T)}{P(w, T)} d\hat{W}(w) - \int_t^T \frac{1}{2} \left(\sigma(r, w) \frac{P_r(w, T)}{P(w, T)} \right)^2 dw \\ &= \ln P(t, T) + \int_t^T r(w) dw + \int_t^T \sigma(r, w) \frac{P_r(w, T)}{P(w, T)} d\hat{W}(w) \\ &\quad - \int_t^T \frac{1}{2} \left(\sigma(r, w) \frac{P_r(w, T)}{P(w, T)} \right)^2 dw \end{aligned} \tag{A3}$$

As a result,

$$(A4) \quad \zeta = \frac{\delta(t, T)}{P(t, T)} = \exp \left[\int_t^T \sigma(r, w) \frac{P_r(w, T)}{P(w, T)} d\hat{W}(w) - \int_t^T \frac{1}{2} \left(\sigma(r, w) \frac{P_r(w, T)}{P(w, T)} \right)^2 dw \right]$$

This implies the Girsanov transformation of the following:

$$(A5) \quad \begin{aligned} d\hat{W}(t) &= d\bar{W}(t) + \sigma(r, t) \frac{P_r(t, T)}{P(t, T)} dt \\ &= \bar{W}(t) + b(t, T) dt \end{aligned}$$

and this completes the proof.

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Figure 1
Futures Prices and Bound Values

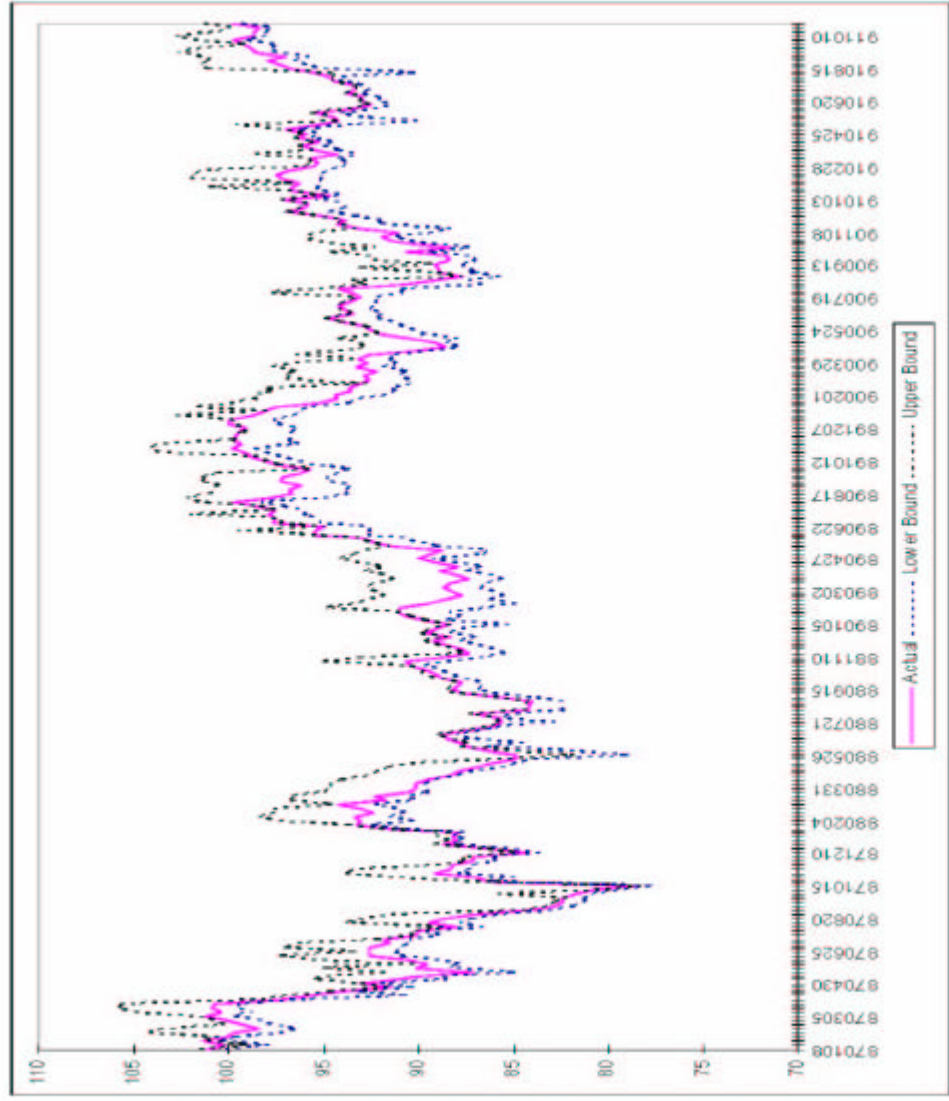


Table 1
Futures Prices and Bound Values

(*)	(+)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
8703	4	100.7032	99.4857	100.5853	100.9902	.5620	1.5045	.0000
8706	13	97.8752	96.1361	100.1316	98.2282	.7404	2.0922	.0000
8709	13	90.7188	89.2040	93.8526	91.2346	.5279	2.0306	.0000
8712	13	84.5458	83.3565	86.3232	85.2438	.8669	1.8872	.0000
8803	13	88.2694	87.5445	89.4068	89.7525	1.4831	2.2080	.0000
8806	13	91.2669	89.9734	95.5133	91.9447	.7118	1.9714	.0000
8809	13	86.6277	84.7276	87.0530	86.6836	.9994	1.9561	.0000
8812	13	87.2692	85.9469	87.5304	88.3137	1.0445	2.3668	.0000
8903	13	89.1226	87.2048	89.9120	89.7004	.9722	2.4955	.0000
8906	13	88.7117	86.5325	92.6549	89.0237	.6311	2.4912	.0000
8909	14	94.9801	92.7628	96.2512	95.3568	.4006	2.5940	.0000
8912	13	97.4713	95.2995	100.3997	97.5632	.1031	2.2637	.0000
9003	12	98.4298	96.1064	100.1697	98.5975	.1704	2.4911	.0000
9006	14	92.2521	90.5492	95.8044	92.9705	.7333	2.4213	.0000
9009	13	93.2383	91.1677	94.0063	93.4697	.5702	2.3020	.0000
9012	13	89.5723	88.0036	92.5594	90.1492	.7717	2.1456	.0000
9103	13	95.1948	93.2753	95.9819	95.2783	.3766	2.0031	.0000
9106	13	96.0026	94.6925	97.5605	96.5474	.5546	1.8549	.0000
9109	13	93.9017	92.7229	94.3216	94.5791	.8984	1.8562	.0000
9112	13	98.1515	96.7921	100.6379	98.3949	.5810	1.6028	.0000
87-91	252	92.4144	90.7584	94.3061	92.9091	.6905	2.1507	.0000

Note: (*) contract month

(†) number of observations

(1) actual futures price

(2) lower bound without the wild card option using Eq 30

(3) cost-of-carry model using Eq 25

(4) futures price with the quality option using Eq 3

(5) total value of timing options (4)-(1)

(6) upper bound of end-of-month and quality options (1)-(2)

(7) upper bound for the wild card option using Eq 21 and the lower bound futures price at time t for $\Phi(t+k)$