Richardson Extrapolation Technique for Pricing American-style Options

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Abstract

In this paper we re-examine the Geske-Johnson (1984) formula and extend the analysis by deriving a modified Geske-Johnson formula that can overcome the possibility of non-uniform convergence encountered in the original Geske-Johnson formula. Furthermore, we propose a numerical method, the repeated Richardson extrapolation, which allows us to estimate the interval of true option values and to determine the number of options needed for an approximation to achieve the desired accuracy. From the simulation results, our modified Geske-Johnson formula is more accurate than the original Geske-Johnson formula. This paper also illustrates that the repeated Richardson extrapolation approach can estimate the interval of true American option values extremely well. Finally we investigate the possibility of combining the Binomial Black and Scholes method proposed by Broadie and Detemple (1996) with the repeated Richardson extrapolation technique.

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1 Introduction

In an important contribution, Geske and Johnson (1984) showed that it was possible to value an American-style option by using a series of options exercisable at one of a finite number of exercise points (known as Bermudan-style options). They employed Richardson extrapolation techniques to derive an efficient computational formula using the values of Bermudan options. The Richardson extrapolation techniques were afterwards used to enhance the computational efficiency and/or accuracy of American option pricing in two directions in the literature. First, one can apply the Richardson extrapolation in the number of time steps of binomial trees to price options. For example, Broadie and Detemple (1996), Tian (1999), and Heston and Zhou (2000) apply a two-point Richardson extrapolation to the binomial option prices. Second, the Richardson extrapolation method has been used to approximate the American option prices with a series of options with an increasing number of exercise points. The existing literature includes Breen (1991) and Bunch and Johnson (1992).

Two problems are recognized to exist with this methodology. First, as pointed out by Omberg (1987), there may in the case of some options be the problem of non-uniform convergence.[1] In general, this arises when a Bermudan option with n exercise points has a value that is less than that of an option with m exercise points, where $m < n$. A second problem with the Geske-Johnson method is

[1] In the Geske-Johnson formula, they defined $P(1)$, $P(2)$ and $P(3)$ as follows: (i) $P(1)$ is a European option, permitting exercise at time T, the maturity date of the option; (ii) $P(2)$ is the value of a Bermudan option, permitting exercise at time $T/2$ or T; (iii) $P(3)$ is the value of a Bermudan option, permitting exercise at time $T/3$, $2T/3$, or T. If the Bermudan option prices converge to the corresponding American option price *uniformly* from below, a Bermudan option with more exercise points must be more valuable than the one with fewer exercise points. In other words, $P(1) < P(2) < P(3) < \cdots$. However, Omberg (1987) showed a plausible example of a non-uniform convergence with a deep-in-the-money put option written on a low volatility, high dividend paying stock going ex-dividend once during the term of the option at time $T/2$. In this case, there is a high probability that the option will be exercised at time $T/2$ immediately after the stock goes ex-dividend. Thus, $P(2)$ could be greater than $P(3)$, and the problem of non-uniform convergence emerges.

that it is difficult to determine the accuracy of the approximation. How many options and/or how many exercise points have to be considered in order to achieve a given level of accuracy?

In this paper we examine these two problems under the Black-Scholes (1973) economy (i.e., asset price following geometric Brownian motion, frictionless markets, continuous trading, constant interest rates, etc.). Following Omberg's (1987) suggestion, we employ geometric exercise points in place of the arithmetic exercise points used by Geske-Johnson. This allows us to derive a modified Geske-Johnson formula which uses only the prices of Bermudan options with uniform convergence property. The numerical results indicate that our modified Geske-Johnson formula is generally more accurate than the original Geske-Johnson formula.

Secondly, we employ a technique known as repeated Richardson approximation.[2] Although the true American option price is generally unknown, Schmidt's (1968) inequality allows us to specify the accuracy of a repeated Richardson approximation. In other words, it helps to determine the smallest value of exercise points (or time steps), n, that can solely be used in an option price approximation for the desired accuracy. Moreover, we also investigate the possibility of combining the Binomial Black and Scholes (hereafter BBS) method proposed by Broadie and Detemple (1996) with the repeated Richardson extrapolation technique.

The plan of this paper is as follows. In section 2 we briefly review the literature on the approximation of American-style option prices with a series of Bermudan options with an increasing number of exercise points. This allows us to specify the incremental contribution of our paper. In section 3 we introduce the repeated Richardson extrapolation technique. Based on geometric exercise points, we apply the repeated Richardson extrapolation to derive a modified Geske-Johnson formula which overcomes the problem of non-uniform convergence encountered in the original Geske-Johnson formula. This paper

[2] We will discuss the repeated Richardson approximation technique in detail in section 3.

also shows how to employ Schmidt's (1968) inequality to determine accuracy of the repeated Richardson extrapolation. In other words, it can tell us how many options and/or exercise points have to be considered to achieve a given level of accuracy. In section 4 we illustrate the methodology with numerical examples. Section 5 concludes the paper.

2 Literature Review

In their original paper, Geske-Johnson show that an American put option can be calculated to a high degree of accuracy using a Richardson approximation. If $P(n)$ is the price of a Bermudan option exercisable at one of n equally-spaced exercise dates, then, for example, using $P(1)$, $P(2)$ and $P(3)$, the price of the American put is approximately

$$P(1,2,3) = P(3) + \frac{7}{2}(P(3) - P(2)) - \frac{1}{2}(P(2) - P(1)), \tag{1}$$

where $P(1,2,3)$ denotes the approximated value of the American option using the values of Bermudan options with 1, 2 and 3 possible exercise points.

In a subsequent contribution, Bunch and Johnson (1992) suggest a modification of the Geske-Johnson method based on the use of an approximation

$$P(1,2) = P^{max}(2) + (P^{max}(2) - P(1)), \tag{2}$$

where $P^{max}(2)$ is the option's value exercisable at one of two points at time, when the exercise points are chosen so as to maximize the option's value. They showed that if the time steps are chosen so as to maximize $P(2)$,[3] then accurate predictions of the American put price can be made with greater computational efficiency than in the case of the original Geske-Johnson method. Moreover, the

[3] Bunch and Johnson suggest that the time of the first exercise point of $P(2)$ can be chosen by examining seven time spaces at $T/8$, $2T/8$, $3T/8$, $4T/8$, $5T/8$, $6T/8$, and $7T/8$ and the time of the second exercisable point is usually allocated at time T, the maturity date of the option.

Bunch and Johnson (1992) method can also avoid the non-uniform convergence problem.

Omberg (1987) and Breen (1991) consider the Geske-Johnson method in the context of binomial computations. Omberg (1987) shows that there may be a problem of non-uniform convergence since $P(2)$ in equation (1) is computed using exercise points at time T and $T/2$, where T is the time to maturity of option, and $P(3)$ is computed using exercise points at time $T/3$, $2T/3$, and T. If the option is a deep-in-the-money put option written on a low volatility, high dividend paying stock going ex-dividend once during the term of the option at time $T/2$, there is a high probability that the option will be exercised at time $T/2$ immediately after the stock goes ex-dividend. Therefore $P(3)$ is not always greater than $P(2)$. Although Breen (1991) also points out the above mentioned problem of non-uniform convergence, he still suggests and tests a binomial implementation of the original Geske-Johnson formula.

It is well known that convergence of a binomial option price to the true price is not uniform, but oscillatory, in the step size (see for example, Broadie and Detemple (1996) and Tian (1999)). The non-uniform convergence limits the use of extrapolation techniques in binomial option pricing models to enhance the rate of convergence. As a result, several papers in the literature have modified the Cox, Ross, and Rubinstein's (1979) (CRR) binomial model to produce uniform convergence. Among them, Broadie and Detemple (1996) propose a method term Binomial Black and Scholes (hereafter BBS) model which gives uniform convergence prices. The BBS method is a modification to the binomial method where the Black-Scholes formula replaces the usual "continuation value" at the time step just before option maturity.

Due to the uniform convergence property of the BBS method, Broadie and Detemple (1996) also suggest a method term Binomial Black and Scholes model with Richardson extrapolation (BBSR). In particular, the BBSR method with n steps computes the BBS prices corresponding to $m = n/2$ steps (say P_m) and n steps (say P_n) and then sets the BBSR price to $P = 2P_n - P_m$. In addition to Broadie and Detemple's (1996) simple two-point Richardson extrapolation, this paper examines the possibility of combining the BBS method with repeated Richardson extrapolation technique.

3 The Repeated Richardson Extrapolation Technique

3.1 The Repeated Richardson Extrapolation Algorithm

Often in numerical analysis, an unknown quantity, a_0 (e.g. the value of an American option in our case), is approximated by a calculable function, $F(h)$, depending on a parameter $h > 0$, such that $F(0) = \lim_{h \to 0} F(h) = a_0$.[4] If we know the complete expansion of the truncation error about the function $F(h)$, then we can perform the repeated Richardson extrapolation technique to approximate the unknown value a_0. Assume that

$$F(h) = a_0 + a_1 h^{\gamma_1} + a_2 h^{\gamma_2} + \ldots + a_k h^{\gamma_k} + O(h^{\gamma_{k+1}}) \tag{3}$$

with known exponents $\gamma_1, \gamma_2, \gamma_3, \cdots$ and $\gamma_1 < \gamma_2 < \gamma_3 \cdots$, but unknown a_1, a_2, a_3, etc., where $O(h^{\gamma_{k+1}})$ denotes a quantity whose size is proportional to $h^{\gamma_{k+1}}$, or possibly smaller. According to Schmidt (1968), we can establish the following algorithm when $\gamma_j = \gamma j$, $j = 1...k$.

Algorithm:

For $i = 1, 2, 3, \cdots$, set $A_{i,0} = F(h_i)$, and compute for $m = 1, 2, 3, \cdots, k-1$.

$$A_{i,0} = F(h_i)$$

[4]The parameter h corresponds to the length between two exercise points of a Bermudan option in the Geske-Johnson approach. The American option value is therefore the limit of a Bermudan option value as h goes to zero.

$$A_{i,m} = A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{(h_i/h_{i+m})^{\gamma} - 1}, \tag{4}$$

where $A_{i,m}$ is an approximate value of a_0 obtained from an m times repeated Richardson extrapolation using step sizes of h_i, h_{i+1}, $\cdots$, h_{i+m}, and $0 < m \leq k - 1$.

The computations can be conveniently set up in the following scheme

h_i	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$\ldots$
h_1	$A_{1,0}$	$A_{1,1}$	$A_{1,2}$	$A_{1,3}$	
h_2	$A_{2,0}$	$A_{2,1}$	$A_{2,2}$		
h_3	$A_{3,0}$	$A_{3,1}$			
h_4	$A_{4,0}$				
$\vdots$					

It should be noted that a repeated Richardson extrapolation will give the same results as those of polynomial Richardson extrapolation methods when the same expansion of the truncation error is used.[5] As an illustration, in the followings we set $\gamma = 1$, $k = 3$ and apply the repeated Richardson extrapolation technique to derive the approximation formulae for American options using arithmetic exercise points (Geske-Johnson formula) and geometric exercise points (modified Geske-Johnson formula), respectively.

3.2 The Geske-Johnson Formulae

In the original Geske-Johnson formulae, they use arithmetic exercise points and set the step sizes as follows: $h_1 = h$, $h_2 = h/2$, $h_3 = h/3$, where h equals the maturity of the option, T. Define $P(1) = A_{1,0}$ (h), the European option

[5]For a rigorous proof of this statement, please refer to Atkinson (1989).

value permitting exercise only at period h, $P(2) = A_{2,0}$ $(h/2)$, the Bermudan-style option value permitting exercise only at period $h/2$ and h, and $P(3) = A_{3,0}$ $(h/3)$, the Bermudan-style option value permitting exercise at period $h/3$, $2h/3$, and h only. By applying the repeated Richardson extrapolation algorithm in equation (4), we can obtain two two-point and one three-point Geske-Johnson formulae, respectively, as follows:

$$A_{1,1} = P(1,2) = 2P(2) - P(1), \tag{5}$$

$$A_{2,1} = P(2,3) = \frac{3}{2}P(3) - \frac{1}{2}P(2), \tag{6}$$

$$A_{1,2} = P(1,2,3) = \frac{9}{2}P(3) - 4P(2) + \frac{1}{2}P(1). \tag{7}$$

It should be noted that $P(1,2)$ and $P(1,2,3)$ are the original Geske and Johnson's two-point and three-point approximation formulae, respectively.

3.3 The Modified Geske-Johnson Formulae

From the previous review of the Geske-Johnson approximation method, we find that it is possible for the condition, $P(1) < P(2) > P(3)$, to occur. Thus, the problem of non-uniform convergence will emerge. To solve this problem, we follow Omberg's suggestion to construct the approximating sequence so that each opportunity set includes the previous one, and therefore is at least as good, by using geometric exercise points $[1, 2, 4, 8, ...]$ generated by successively doubling the number of uniformly-spaced exercise dates, rather than the arithmetic exercise points $[1, 2, 3, 4, ..]$ employed by Geske-Johnson.

If we use geometric exercise points employed in the modified Geske-Johnson formula, we can set the time steps as follows: $h_1 = h$, $h_2 = h/2$, $h_3 = h/4$, where h equals the maturity of the option, T. Define $P(1) = A_{1,0}$ (h), the European option value permitting exercise only at period h, $P(2) = A_{2,0}$ $(h/2)$, the Bermudan-style option value permitting exercise only at period $h/2$ and h, and $P(4) = A_{3,0}$ $(h/4)$, the Bermudan-style option value permitting exercise at period $h/4$, $2h/4$, $3h/4$, and h only. Again we can apply the repeated Richardson extrapolation algorithm to derive two two-point and one three-point modified Geske-Johnson formulae, respectively, as follows:

$$A_{1,1} = P(1,2) = 2P(2) - P(1), \tag{8}$$

$$A_{2,1} = P(2,4) = 2P(4) - P(2), \tag{9}$$

$$A_{1,2} = P(1,2,4) = \frac{8}{3}P(4) - 2P(2) + \frac{1}{3}P(1). \tag{10}$$

Because we use geometric exercise points, we can ensure that $P(4) \geq P(2) \geq P(1)$ always holds in equations (8) to (10). The reason for this is that the exercise points of $P(4)$ include all the exercise points of $P(2)$, while the exercise points of $P(2)$ include all the exercise points of $P(1)$. Thus, the modified Geske-Johnson formula is able to overcome the shortcomings of non-uniform convergence encountered in the original Geske-Johnson formula.[6]

3.4 The Error Bounds and Predictive Intervals of American Option Values

One specific advantage in using a repeated Richardson extrapolation is that we can obtain the error bounds of the approximation and thus predict the interval of the true American option values. In other words, the repeated Richardson extrapolation technique allows us to determine the accuracy of the approximation and also how many options or how many exercise points have to be considered in order to achieve a given accuracy. This can be done by applying the Schmidt's (1968) inequality.

[6] However we need a four dimensional normal integral while Geske and Johnson only need a three dimensional normal integral.

Schmidt's Inequality

Schmidt (1968) shows that it always holds

$$\Big|A_{i,m+1} - F(0)\Big| \leq \Big|A_{i,m+1} - A_{i,m}\Big|, \tag{11}$$

when i is sufficiently large[7] and m is under the constraint, $0 < m \leq k - 1$.[8] Here, $F(0)$ is the true American option value, and $A_{i,m}$ is the approximate value of $F(0)$ obtained from using the m-times repeated Richardson extrapolation.

When Schmidt's inequality holds (i.e. when i is sufficiently large), we know that (i) the error of the approximation $A_{i,m+1}$ is smaller than $\Big|A_{i,m+1} - A_{i,m}\Big|$, (ii) if the desired accuracy is ϵ and i and m are the smallest integers that $\Big|A_{i,m+1} - A_{i,m}\Big| \leq \epsilon$ holds, then the approximation $A_{i,m+1}$ is accurate enough for the desired accuracy. Furthermore we know that $m + 2$ Bermudan options with step sizes, h_i, h_{i+1}, $\cdots$, h_{i+m+1}, have to be considered to achieve the desired accuracy. (iii) The true value of the American option is within the range $\Big(A_{i,m+1} - \Big|A_{i,m+1} - A_{i,m}\Big|,\ \ A_{i,m+1} + \Big|A_{i,m+1} - A_{i,m}\Big|\Big)$.

[7] In the literature, mathematicians note that it is very difficult to say how large i must be in order to ensure that $A_{i,m}$ and $U_{i,m}$ ($U_{i,m}$ is defined in Appendix) are the upper or lower bound of $F(0)$. However, they suggest that, for practical purpose, the extrapolation should be stopped "if a finite number of $A_{i,m}$ and $U_{i,m}$ decrease or increase monotonically, and if $\Big|A_{i,m} - U_{i,m}\Big|$ is small enough for accuracy." Apart from using the above suggestion, from Tables 4 and 5 we found out that when $i = 2$ and $m = 1$, $m = 2$, or $m = 3$, there are only a very low percentage violate the inequality. However, the violation of error boundaries is not very significant. Thus, we can ignore them.

[8] The proof of this inequality is presented in the Appendix following Schmidt (1968).

4 Numerical Analysis

4.1 Choosing the Benchmark Method

The accurate American option values are generally unknown and are usually estimated using CRR binomial method with a very large number (say $10,000$) of time steps. However we need very accurate American option values for the following analyses. One of the most accurate binomial method in the literature is the BBSR method proposed by Broadie and Detemple (1996). Therefore we compare the accuracy of the CRR and BBSR models to decide the benchmark method.

The accuracy of the CRR and BBSR models is examined for European put options because their accurate values (Black-Scholes) are known. The root-mean-squared (hereafter RMS) relative error is used as the measure of accuracy. The RMS error is defined by

$$RMS = \sqrt{\frac{1}{j}\sum_{k=1}^{j} {e_k}^2}, \tag{12}$$

where $e_k = (P_k^* - P_k)/P_k$ is the relative error, P_k is the true option price (Black-Scholes), P_k^* is the estimated option price using each method with 10800 time steps, and j is the number of options considered. Following Broadie and Detemple (1996), we price a large set ($j = 243$) of options with practical parameters: $K = 100$; $S = 90$, 100, 110; $\sigma = 0.2$, 0.3, 0.4; $T = 0.25$, 0.5, 1.0; $r = 0.03$, 0.08, 0.13; and $q = 0$, 0.02, 0.04.

It is clear from Table 1 that the RMS relative error of the BBSR method (0.0000448%) is far smaller than that of the CRR method (0.00301%). Our result is consistent with the findings of Broadie and Detemple (1996). Therefore, we will use the BBSR method with 10800 steps to calculate benchmark prices of American options in the following analyses.

4.2 The Accuracy of the Geske-Johnson Formulae vs. the Modified Geske-Johnson Formulae

In this section we compare the accuracy of the Geske-Johnson formulae with that of the Modified Geske-Johnson formulae. As an illustration, we first study the accuracy of only a three-point approximation for both methods. In other words, we investigate the accuracy of $P(1, 2, 3)$ and $P(1, 2, 4)$ in equations (7) and (10). To evaluate $P(2)$, $P(3)$, and $P(4)$, we implement two-, three-, and four-dimensional normal integrals, respectively, using the IMSL subroutines for FORTRAN language.

In Table 2, we show the accuracy of a three-point Geske-Johnson formula and that of a three-point modified Geske-Johnson formula. It is evident from Table 2 that the modified Geske-Johnson formula generally produces more accurate approximation than the original Geske-Johnson formula. From Table 2, we find that the modified Geske-Johnson formula is more accurate for 21 out of 27 options.

We now turn to a detail analysis of the accuracy of the Richardson extrapolation for the number of exercise points to estimate American option values. Both arithmetic and geometric exercise points are examined. The analysis is based on five (i.e. $i = 1, 2, \cdots, 5$.) different step sizes and up to four repeated times in the Richardson extrapolation. As before we use 243 options to conduct the analysis and use the RMS relative error as the measure of accuracy.

Table 3 shows the RMS relative errors in pricing American options using the repeated Richardson extrapolation with arithmetic and geometric exercise points. The true values of all Bermudan options are estimated by the BBSR method with $10,800$ steps.[9] The results indicate that the pricing errors of geometric

[9]Although the analytic solutions are available for $P(5)$, $P(8)$, and $P(16)$, however their evaluations involve high dimensional numerical integration. Therefore we use the BBSR method with $10,800$ steps to calculate the accurate values for all Bermudan options for consistency.

exercise points are smaller than that of arithmetic exercise points. This finding supports that a Richardson extrapolation with geometric exercise points can avoid the problem of non-uniform convergence. Moreover, the repeated Richardson extrapolation technique can further reduce the pricing errors. In other words, an $(n+1)$-point Richardson extrapolation generally produces more accurate prices than an n-point Richardson extrapolation. For example, Panel B shows that the RMS relative errors of $A_{1,2}$ (obtained from a three-point Richardson extrapolation of $P(1)$, $P(2)$, and $P(4)$) is 0.346 %, which is smaller than that (1.061 %) of $A_{1,1}$ (obtained from a two-point Richardson extrapolation of $P(1)$ and $P(2)$) and that (0.427 %) of $A_{2,1}$ (obtained from a two-point Richardson extrapolation of $P(2)$ and $P(4)$).

4.3 The Validity of Schmidt's Inequality

One specific advantage of the repeated Richardson extrapolation is that it allows us to specify the accuracy of an approximation to the unknown true option price. That is, the Schmidt's inequality can be used to predict tight upper and lower bounds (with desired tolerable errors) of the true option values. We test the validity of the Schmidt's inequality over 243 options for both geometric and arithmetic exercise points in Tables 4 and 5, respectively. The denominator represents the number of options whose price estimates match $\left|A_{i,m+1} - A_{i,m}\right| <$ the desired errors, and the numerator is the number of options whose price estimates match $\left|A_{i,m+1} - F(0)\right| <$ the desired errors and $\left|A_{i,m+1} - A_{i,m}\right| <$ the desired errors.

The results in Tables 4 and 5 indicate that increasing i or m will increase the number of price estimates with errors less than the desired accuracy. It is also clear that the Schmidt inequality is seldom violated especially when i or m is large ($i = 3, 4$ and $m = 2, 3$). For example, when $i = m = 2$, 228 out of 243 option price estimates have errors smaller than 0.2% of the European option value, and 225 out of these 228 option price estimates satisfy Schmidt's inequality. Moreover, the findings support that the repeated Richardson extrapolation with geometric exercise points works better than with arithmetic exercise points. This supports the previous result that a Richardson extrapolation with geometric exercise points can avoid the problem of non-uniform convergence.

4.4 The Accuracy of the BBS Method with Repeated Richardson Extrapolation Techniques

In this subsection we investigate the possibility of combining the BBS method with the repeated Richardson extrapolation technique. We apply the BBS method with a repeated Richardson extrapolation in *number of time steps* to price European put options, because the true prices are easy to calculate. Both the arithmetic and geometric time steps are analyzed. As before, we choose 243 options with practical parameters: $K = 100$; $S = 90$, 100, 110; $\sigma = 0.2$, 0.3, 0.4; $T = 0.25$, 0.5, 1.0; $r = 0.03$, 0.08, 0.13; and $q = 0$, 0.02, 0.04.

Many points can be drawn from Table 6. First, it is clear from the third column of Table 6 that the pricing error of an N-step BBS model for standard options is at the rate of $O(1/N)$. In contrast, Heston and Zhou (2000) show that the pricing error of an N-step CRR model fluctuates between the rate of $O(1/\sqrt{N})$ and $O(1/N)$. As a result, the BBSR method with geometric time steps produces very accurate prices for European options (see the fourth column of Panel B in Table 6). Second, the pricing errors from geometric time steps are far smaller than that of arithmetic time steps. Third, Table 6 reveals that the repeated Richardson extrapolation in time steps cannot further improve the accuracy. For example, Panel B shows that the pricing error of $A_{4,1}$ (obtained from a two-point Richardson extrapolation of BBS prices with 160 and 320 steps) is actually smaller than that of $A_{3,2}$ (obtained from a three-point Richardson extrapolation of BBS prices with 80, 160, and 320 steps).

5 Conclusion

In this paper we re-examine the original Geske-Johnson formula. We first extend the analysis by deriving a modified Geske-Johnson formula which is able to overcome the possibility of non-uniform convergence encountered in the original Geske-Johnson formula. Another contribution of this paper is that we propose a numerical method which can estimate the predicted intervals of the true option values when the accelerated binomial option pricing models are used to value the American-style options.

The findings are summarized as follows: (i) The modified Geske-Johnson formula is a better approximation of American option price than the original Geske-Johnson formula. This is not surprised because the modified Geske-Johnson formula can overcome the non-uniform convergence problem. (ii) Using Schmidt's inequality, we are able to obtain the intervals of the true American option values. This helps to specify the accuracy of an approximation to the unknown true option price and to determine the number of options that can solely be used in an option price approximation. This article probably is the first one to discuss how to get the predicted intervals of the true option values in the finance literature. We believe that the repeated Richardson method will be very useful for practitioners to predict the intervals of the true option values. (iii) The Richardson extrapolation approach can improve the computational accuracy for the BBS method proposed by Broadie and Detemple (1996), while two- or more- times repeated Richardson extrapolation technique cannot.

Appendix: The Proof of Schmidt's Inequality

In this appendix we prove that $\left|A_{i,m+1} - F(0)\right| \leq \left|A_{i,m+1} - A_{i,m}\right|$ is always true when i is sufficiently large and m is under the constraint, $0 < m \leq k-1$, where k is the order of powers of the expansion of truncation errors. Let $F(h)$ be the appropriate solution gained through discretization for a problem. We assume that $F(h)$ can be developed for the parameter $h > 0$

$$F(h) = a_0 + a_1 h^{\gamma_1} + a_2 h^{\gamma_2} + \ldots + a_k h^{\gamma_k} + O(h^{\gamma_{k+1}}), \tag{13}$$

where $\gamma_1 < \gamma_2 < \gamma_3 < .... < \gamma_{k+1}$. The solution of the original problem is $F(0) = \lim_{h \to 0} = a_0$.

Schmidt (1968) shows that, when $\gamma_k = \gamma k + \delta$ and $h_{i+1}/h_i \leq \rho \leq 1$ (ρ is a constant and $0 \leq \rho \leq 1$), iterative extrapolation can be carried out according to the following procedure

$$\begin{aligned} A_{i,0} &= F(h_i) \\ H_{i,0} &= {h_i}^{-\delta}, \\ A_{i,m} &= A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{D_{i,m-1} - 1}, \\ H_{i,m} &= H_{i+1,m-1} + \frac{H_{i+1,m-1} - H_{i,m-1}}{[h_i/h_{i+m}]^{\gamma} - 1}, \end{aligned} \tag{14}$$

where

$$D_{i,m} = \frac{{h_i}^{\gamma} H_{i+1,m-1}}{h_{i+m}^{\gamma} H_{i,m-1}},$$

and $0 < m \leq k-1$.

If δ is equal to zero (i.e. $\gamma_k = \gamma k$), then $H_{i,m}$ is equal to one. Thus, equation (14) can be reduced to the following equation

$$\begin{aligned} A_{i,0} &= F(h_i) \\ A_{i,m} &= A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{D_{i,m-1} - 1}, \end{aligned} \tag{15}$$

where

$$D_{i,m} = [h_i/h_{i+m}]^{\gamma}.$$

Schmidt defined $U_{i,m}$ as the following

$$U_{i,m} = (1+\beta)A_{i+1,m} - \beta A_{i,m}, \tag{16}$$

where

$$\beta = 1 + \frac{2}{[h_i/h_{i+m+1}]^\gamma - 1} = 1 + \frac{2}{D_{i,m+1} - 1}.$$

According to the proof of theorem 2 in Schmidt's paper, we can get equation (17) when i is sufficiently large, m is under the constraint, $0 < m \leq k-1$ and a_{m+1} $(m = 1, ..k-1)$ is not equal to zero, i.e.

$$\begin{aligned} A_{i,m} \leq F(0) \leq U_{i,m}, \\ \text{or } \; U_{i,m} \leq F(0) \leq A_{i,m}. \end{aligned} \tag{17}$$

This is equivalent to

$$\Big|[A_{i,m} + U_{i,m}]/2 - F(0)\Big| \leq \frac{1}{2}\Big|U_{i,m} - A_{i,m}\Big|. \tag{18}$$

Rearranging the definition of $U_{i,m}$ in equation (16), we obtain the following equation

$$\frac{1}{2}(A_{i,m} + U_{i,m}) = \frac{1}{2}(1+\beta)A_{i+1,m} + \frac{1}{2}(1-\beta)A_{i,m} \tag{19}$$

Furthermore, from the definition of β in equation (16), we are able to get the following relationship

$$\begin{aligned} 1+\beta &= 2\left(1 + \frac{1}{D_{i,m+1} - 1}\right), \\ 1-\beta &= \frac{-2}{D_{i,m+1} - 1}. \end{aligned} \tag{20}$$

Substituting equation (20) into equation (19) and referring to equation (15), we obtain

$$\frac{1}{2}(A_{i,m} + U_{i,m}) = A_{i,m+1}. \tag{21}$$

Similarly, we also can acquire the following relationship

$$\frac{1}{2}(U_{i,m} - A_{i,m}) = A_{i,m+1} - A_{i,m}. \tag{22}$$

Finally, substituting equations (21) and (22) into equation (18), we obtain Schmidt's inequality

$$\Big|A_{i,m+1} - F(0)\Big| \leq \Big|A_{i,m+1} - A_{i,m}\Big|. \tag{23}$$

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Table 1:
The root-mean-squared (RMS) relative errors using the binomial and BBSR methods to price European options

binomial	BBSR
3.01E-05	4.48E-07

The options are European put options. The root-mean-squared relative errors are defined as follows:

$$RMS = \sqrt{\frac{1}{j}\sum_{k=1}^{j} e_k^2},$$

where $e_k = (P_k^* - P_k)/P_k$ is the relative error, P_k is the true option price (Black-Scholes), and P_k^* is the estimated option price. The number of steps in each method is 10,800. The strike price (K) is 100. There are 243 options with practical parameters: S =90, 100, 110; σ=0.2, 0.3, 0.4; T=0.25, 0.5, 1 years; r=3, 8, 13%; and q=0, 2, 4%.

Table 2:
Valuation of American Put Options

(1) K	(2) σ	(3) T	(4) $P(1)$	(5) $P(\infty)$	(6) $P(1,2,3)$	$\frac{(6)-(5)}{(5)}$ %	(7) $P(1,2,4)$	$\frac{(7)-(5)}{(5)}$ %
35	0.2	0.0833	0.0062	0.0062	0.0062	-0.484%	0.0062	-0.416%
35	0.2	0.3333	0.1960	0.2004	0.1999	-0.255%	0.1999	-0.234%
35	0.2	0.5833	0.4170	0.4329	0.4326	-0.066%	0.4325	-0.093%
40	0.2	0.0833	0.8404	0.8523	0.8521	-0.027%	0.8522	-0.015%
40	0.2	0.3333	1.5222	1.5799	1.5760	-0.251%	1.5772	-0.174%
40	0.2	0.5833	1.8813	1.9906	1.9827	-0.395%	1.9847	-0.297%
45	0.2	0.0833	4.8399	5.0000	4.9969	-0.062%	4.9973	-0.055%
45	0.2	0.3333	4.7805	5.0884	5.1053	0.332%	5.1027	0.281%
45	0.2	0.5833	4.8402	5.2671	5.2893	0.421%	5.2850	0.340%
35	0.3	0.0833	0.0771	0.0775	0.0772	-0.273%	0.0773	-0.219%
35	0.3	0.3333	0.6867	0.6976	0.6973	-0.049%	0.6972	-0.063%
35	0.3	0.5833	1.1890	1.2199	1.2199	-0.005%	1.2197	-0.020%
40	0.3	0.0833	1.2991	1.3102	1.3103	0.010%	1.3103	0.007%
40	0.3	0.3333	2.4276	2.4827	2.4801	-0.105%	2.4811	-0.065%
40	0.3	0.5833	3.0636	3.1698	3.1628	-0.221%	3.1651	-0.149%
45	0.3	0.0833	4.9796	5.0598	5.0631	0.065%	5.0623	0.049%
45	0.3	0.3333	5.5290	5.7058	5.7019	-0.068%	5.7017	-0.071%
45	0.3	0.5833	5.9725	6.2438	6.2368	-0.112%	6.2367	-0.113%
35	0.4	0.0833	0.2458	0.2467	0.2463	-0.163%	0.2464	-0.128%
35	0.4	0.3333	1.3298	1.3462	1.3461	-0.004%	1.3459	-0.021%
35	0.4	0.5833	2.1129	2.1550	2.1553	0.011%	2.1550	0.000%
40	0.4	0.0833	1.7579	1.7685	1.7688	0.017%	1.7687	0.010%
40	0.4	0.3333	3.3338	3.3877	3.3863	-0.041%	3.3869	-0.022%
40	0.4	0.5833	4.2475	4.3529	4.3475	-0.123%	4.3496	-0.077%
45	0.4	0.0833	5.2362	5.2870	5.2848	-0.041%	5.2851	-0.036%
45	0.4	0.3333	6.3769	6.5100	6.5015	-0.130%	6.5035	-0.100%
45	0.4	0.5833	7.1657	7.3832	7.3696	-0.184%	7.3726	-0.144%

The first four columns are from Table 1 of Geske and Johnson (1984). Columns (1) to (3) represent the parameter input for K, the option strike price, σ, the volatility of the underlying asset, and T, the time to expiration. Column (4) shows the European put option values, $P(1)$. Column (5) shows the benchmark values of American put options obtained from the BBSR method with 10,800 steps, $P(\infty)$. Column (6) shows the three-point GJ American put option values, $P(1,2,3)$, using $P(1)$, $P(2)$, and $P(3)$. Column (7) reports the results of our three-point modified GJ approximation formula, $P(1,2,4)$, using $P(1)$, $P(2)$, and $P(4)$. The risk free rate r is 0.05 and the initial stock price S is 40.

Table 3:
The RMS relative errors using the repeated Richardson extrapolation in number of exercisable points to estimate American option values

Panel A: arithmetic exercise points							
i	$A_{i,0}$	h_i	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$A_{i,4}$
1	P(1)=$A_{1,0}$	$h_1 = h$	0.09184	0.01061	0.00432	0.00200	0.00122
2	P(2)=$A_{2,0}$	$h_2 = h/2$	0.04678	0.00537	0.00215	0.00121	
3	P(3)=$A_{3,0}$	$h_3 = h/3$	0.03160	0.00327	0.00137		
4	P(4)=$A_{4,0}$	$h_4 = h/4$	0.02392	0.00226			
5	P(5)=$A_{5,0}$	$h_5 = h/5$	0.01925				
Panel B: geometric exercise points							
i	$A_{i,0}$	h_i	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$A_{i,4}$
1	$P(1) = A_{1,0}$	$h_1 = h$	0.09184	0.01061	0.00346	0.00116	0.00051
2	$P(2) = A_{2,0}$	$h_2 = h/2$	0.04678	0.00427	0.00122	0.00053	
3	$P(4) = A_{3,0}$	$h_3 = h/4$	0.02392	0.00159	0.00057		
4	$P(8) = A_{4,0}$	$h_4 = h/8$	0.01215	0.00074			
5	$P(16) = A_{5,0}$	$h_5 = h/16$	0.00614				

The options are American put options. The RMS relative errors are defined as follows:

$$RMS = \sqrt{\frac{1}{j}\sum_{k=1}^{j} e_k^2},$$

where $e_k = (P_k^* - P_k)/P_k$ is the relative error, P_k is the true American option price (estimated by the BBSR method with 10,800 steps), and P_k^* is the estimated option price. The true values of $P(1)$, $P(2)$,..., $P(5)$ in the arithmetic case and the true values of $P(1)$, $P(2)$,..., $P(16)$ in the geometric case are estimated by the BBSR method with 10,800 steps. The strike price (K) is 100. There are 243 options with practical parameters: S=90, 100, 110;σ=0.2, 0.3, 0.4; T=0.25, 0.5, 1 years; r=3, 8, 13%; and q=0, 2, 4%.

Table 4:
The Validity of the Schmidt Inequality when the Repeated Richardson Extrapolation Is Used in Geometric Exercise Points

Panel A: desired error=1% $*P(1)$				
i \ $(m,m+1)$	(0,1)	(1,2)	(2,3)	(3,4)
1	$\frac{61}{66}$(92.4%)	$\frac{173}{178}$(97.2%)	$\frac{238}{238}$(100%)	$\frac{243}{243}$(100%)
2	$\frac{74}{74}$(100%)	$\frac{235}{235}$(100%)	$\frac{243}{243}$(100%)	
3	$\frac{109}{109}$(100%)	$\frac{243}{243}$(100%)		
4	$\frac{171}{171}$(100%)			
Panel B: desired error=0.2% $*P(1)$				
i \ $(m,m+1)$	(0,1)	(1,2)	(2,3)	(3,4)
1	$\frac{27}{63}$(75%)	$\frac{43}{60}$(71.7%)	$\frac{149}{156}$(95.5%)	$\frac{227}{228}$(99.6%)
2	$\frac{29}{29}$(100%)	$\frac{118}{120}$(98.3%)	$\frac{225}{228}$(98.7%)	
3	$\frac{36}{36}$(100%)	$\frac{213}{214}$(99.5%)		
4	$\frac{53}{53}$(100%)			
Panel C: desired error=0.05% $*P(1)$				
i \ $(m,m+1)$	(0,1)	(1,2)	(2,3)	(3,4)
1	$\frac{18}{21}$(85.7%)	$\frac{21}{27}$(77.8%)	$\frac{84}{94}$(89.4%)	$\frac{150}{158}$(94.9%)
2	$\frac{16}{16}$(100%)	$\frac{45}{48}$(93.8%)	$\frac{167}{181}$(92.3%)	
3	$\frac{21}{21}$(100%)	$\frac{123}{124}$(99.2%)		
4	$\frac{26}{26}$(100%)			

The denominator represents the number of option price estimates that match $|A_{i,m+1} - A_{i,m}| <$ the desired errors, and the numerator is the number of option price estimates that match $|A_{i,m+1} - F(0)| <$ the desired errors and $|A_{i,m+1} - A_{i,m}| <$ the desired errors. The number in the bracket represents the percentage that the Schmidt inequality is sustained.

Table 5:
The Validity of the Schmidt Inequality when the Repeated Richardson Extrapolation Is Used in Arithmetic Exercise Points

Panel A: desired error=1% $*P(1)$				
i \ $(m, m+1)$	(0,1)	(1,2)	(2,3)	(3,4)
1	$\frac{61}{66}$(92.4%)	$\frac{170}{180}$(94.4%)	$\frac{233}{234}$(99.6%)	$\frac{243}{243}$(100%)
2	$\frac{78}{78}$(100%)	$\frac{288}{289}$(99.6%)	$\frac{242}{242}$(100%)	
3	$\frac{97}{97}$(100%)	$\frac{240}{240}$(100%)		
4	$\frac{109}{109}$(100%)			
Panel B: desired error=0.2% $*P(1)$				
i \ $(m, m+1)$	(0,1)	(1,2)	(2,3)	(3,4)
1	$\frac{27}{36}$(75%)	$\frac{44}{62}$(71%)	$\frac{128}{142}$(90.1%)	$\frac{180}{188}$(95.7%)
2	$\frac{29}{29}$(100%)	$\frac{90}{103}$(87.4%)	$\frac{185}{192}$(96.4%)	
3	$\frac{34}{34}$(100%)	$\frac{143}{151}$(94.7%)		
4	$\frac{38}{38}$(100%)			
Panel C: desired error=0.05% $*P(1)$				
i \ $(m, m+1)$	(0,1)	(1,2)	(2,3)	(3,4)
1	$\frac{18}{21}$(85.7%)	$\frac{19}{27}$(70.4%)	$\frac{54}{60}$(90%)	$\frac{81}{105}$(77.1%)
2	$\frac{16}{16}$(100%)	$\frac{36}{43}$(83.7%)	$\frac{100}{116}$(86.2%)	
3	$\frac{19}{19}$(100%)	$\frac{53}{63}$(92.1%)		
4	$\frac{21}{21}$(100%)			

The denominator represents the number of option price estimates that match $|A_{i,m+1} - A_{i,m}| <$ the desired errors, and the numerator is the number of option price estimates that match $|A_{i,m+1} - F(0)| <$ the desired errors and $|A_{i,m+1} - A_{i,m}| <$ the desired errors. The number in the bracket represents the percentage that the Schmidt inequality is sustained.

Table 6:
The RMS relative errors using the BBS with the repeated Richardson extrapolation in number of time steps to price European options

Panel A: arithmetic time steps						
i	number of steps	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$A_{i,4}$
1	20	0.00430	0.00020	0.00095	0.00076	0.00058
2	40	0.00220	0.00073	0.00078	0.00059	
3	60	0.00146	0.00076	0.00061		
4	80	0.00111	0.00065			
5	100	0.00088				
Panel B: geometric time steps						
i	number of steps	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$A_{i,4}$
1	20	0.00430	0.00020	0.00014	7.41E-05	4.23E-05
2	40	0.00220	7.44E-05	5.10E-05	3.64E-05	
3	80	0.00111	2.60E-05	2.74E-05		
4	160	0.00056	1.60E-05			
5	320	0.00028				

The options are European put options. The RMS relative errors are defined as follows:

$$RMS = \sqrt{\frac{1}{j}\sum_{k=1}^{j} e_k^2},$$

where $e_k = (P_k^* - P_k)/P_k$ is the relative error, P_k is the true option price (Black-Scholes), and P_k^* is the estimated option price. The strike price (K) is 100. There are 243 options with practical parameters: S=90, 100, 110; σ=0.2, 0.3, 0.4; T=0.25, 0.5, 1 years, r=3, 8, 13%; and q=0, 2, 4%. Note that $A_{i,0}$, $A_{i,1}$ correspond to the BBS and BBSR methods of Broadie and Detemple (1996), respectively.